Hyperbolic graph generator

August 19, 2014

Contents

1 Introduction

This software implements and extends the network model described in [1]. Embedded in the hyperbolic plane, these networks naturally exhibit two common properties of real-world networks, namely power-law node degree distribution and strong clustering. Moreover, other well-known graph ensembles, such as the soft configuration model (SCM), Erdős-Rényi (ER) graphs, or random geometric graphs (RGGs), appear as degenerate regimes in the model. Table 1 shows all the regimes in the model. Each of them is defined by two parameters: γ, which is the expected exponent of the degree distribution, and temperature T, the parameter controlling the strength of clustering in the network.

γ \ T	0	$(0, \infty)$	∞
$[2, \infty)$	Hyperbolic RGGs	Soft hyperbolic RGGs	Soft configuration model
∞	Spherical RGGs	Soft spherical RGGs	Erdős-Rényi

Table 1: Regimes in the model.

The user selects a regime by specifying appropriate values of γ and T. If $\gamma > 10$, then γ is considered infinite. If $T > 10$, then T is considered infinite. The threshold values of these infinities can be modified by changing the HG_INF_TEMPERATURE and HG_INF_GAMMA definitions in the package.

The full list of input parameters and their default values is as follows:

N - Number of nodes; default $N = 1000$.

$\bar{k}$ - Expected average degree; default $\bar{k} = 10$.

γ - Expected power-law exponent of the degree distribution; default $\gamma = 2$.

T - Temperature; default $T = 0$.

ζ - Square root of the hyperbolic plane curvature $K = -\zeta^2$; default $\zeta = 1$.

s - Random seed; default $s = 1$.

If $T \geq$ HG_INF_TEMPERATURE and $\gamma <$ HG_INF_GAMMA, then the ζ parameter is interpreted as the $\eta = \zeta/T$ parameter in the soft configuration model regime. Its default value is $\eta = 1$.

Given the input parameters, the graph generation process consists of three steps:

1. Compute the internal parameters, such as the radius R of the hyperbolic disk occupied by nodes, as functions of the input parameter values, Sections 3,4.

2. Assign to all nodes their angular and radial coordinates on the hyperbolic plane, Section 2.

3. Connect each node pair by an edge with probability (the connection probability), which is a function of the coordinates of the two nodes, Sections 3,4.

2 Sampling node coordinates

The assignment of node coordinates is done as follows in all the six regimes.

2.1 Angular coordinates

Angular coordinates θ of nodes are assigned by sampling them uniformly at random from interval $[0, 2\pi)$, i.e., the angular node density is uniform $\rho(\theta) = 1/(2\pi)$.

2.2 Radial coordinates

Radial coordinates $r \in [0, R]$, where R is the radius of the hyperbolic disk, are sampled from the following distribution, which is nearly exponential with exponent $\alpha > 0$,

$$\rho(r) = \alpha \frac{\sinh \alpha r}{\cosh \alpha R - 1} \approx \alpha e^{\alpha(r-R)} \sim e^{\alpha r}. \tag{1}$$

The calculation of internal parameter R is described in detail below; it is different in different regimes. Internal parameter α depends on the expected power-law exponent γ and on the curvature of the hyperbolic space $\zeta = \sqrt{-K}$. For temperatures $T \leq 1$, this relationship is given by

$$\gamma = 2\frac{\alpha}{\zeta} + 1, \tag{2}$$

while for $T > 1$ it becomes

$$\gamma = 2\frac{\alpha}{\zeta} T + 1. \tag{3}$$

To sample radial coordinates r according to the distribution in Eq. (1), the inverse transform sampling is used: first a random value U_i is sampled from the uniform distribution on $[0, 1]$, and then the radial coordinate of node i is set to

$$r_i = \frac{1}{\alpha} \operatorname{acosh}\left(1 + (\cosh \alpha R - 1)\, U_i\right), \quad \text{for } i = 1, .., N. \tag{4}$$

3 Finite $\gamma \geq 2$

3.1 $T \in (0, \infty)$: Soft hyperbolic random geometric graphs

This is the most general regime in the model, from which all other regimes can be obtained as limit cases. The connection probability in this case is

$$p(x) = \frac{1}{1 + e^{\beta(\zeta/2)(x-R)}}, \tag{5}$$

where $\beta = 1/T$, and R is the radius of the hyperbolic disk occupied by nodes. The hyperbolic distance x between two nodes at polar coordinates (r, θ) and

(r', θ') is given by

$$x = \frac{1}{\zeta} \text{arccosh} \left(\cosh \zeta r \cosh \zeta r' - \sinh \zeta r \sinh \zeta r' \cos \Delta\theta \right), \tag{6}$$

where $\Delta\theta \; = \pi - |\pi - |\theta - \theta'||$ is the angular distance between the nodes. To calculate the expected degree of a node at radial coordinate r, without loss of generality its angular coordinate can be set to zero, $\theta = 0$, so that its expected degree can be written as

$$\bar{k}(r) = \frac{N}{\pi} \int_0^R \rho(r') \int_0^\pi p(x) \, d\theta' dr'. \tag{7}$$

The expected average degree in the network is then

$$\bar{k} = \int_0^R \rho(r) \, \bar{k}(r) dr = \frac{N}{\pi} \int_0^R \rho(r) \int_0^R \rho(r') \int_0^\pi p(x) \, d\theta' dr' dr. \tag{8}$$

Given user-specified values of input parameters N, $\beta = 1/T$, ζ and $\bar{k}$, the last equation is solved for R using the bisection method in combination with numeric evaluation of the integrals in the equation. The MISER Monte Carlo algorithm from the GSL library is used to compute the multidimensional integral in Eq. (8). The iterative bisection procedure to find R stops when the difference between the value of the computed integral in Eq. (8) and the target value of $\bar{k}$ is smaller than a threshold that is set to 10^{-2} by default.

3.2 Limit $T \to 0$: Hyperbolic random geometric graphs

In the $T \to 0$ ($\beta \to \infty$) limit, the connection probability in Eq. (5) becomes

$$p(x) = \Theta(R - x), \tag{9}$$

where $\Theta(x)$ is the Heaviside step function, meaning that two nodes are connected if the hyperbolic distance x between them is less than R, or they are not connected otherwise. The expected average degree in the network is given by the same Eq. (8), but with $p(x)$ in the last equation. The value of R is determined using the same procedure as in Section 3.1. The only difference is that function $p(x)$ is given by Eq. (9).

3.3 Limit $T \to \infty$: Soft configuration model

According to Eq. (3), in the $T \to \infty$ limit with finite α, to have finite γ, curvature should also go to infinity, $\zeta \to \infty$, such that $\eta = \zeta/T$ is finite, and instead of Eq. (3) one gets

$$\gamma = 2\frac{\alpha}{\eta} + 1. \tag{10}$$

More importantly, one can show that as a result of $\zeta \to \infty$, the expression for hyperbolic distance x between two nodes in Eq. (6) degenerates to

$$x = r + r', \tag{11}$$

meaning that in the $T \to \infty$ regime the angular coordinates are completely ignored. The connection probability becomes

$$p(r, r') = \frac{1}{1 + e^{(\eta/2)(r+r'-R)}}, \tag{12}$$

and the expected average degree in the network is

$$\bar{k} = N \int_0^R \rho(r) \int_0^R \rho(r')\, p(r, r')\, dr' dr. \tag{13}$$

The value of R is determined using the same combination of the bisection method and numeric integration as in the previous section, except it is applied to Eq. (13).

4 Infinite $\gamma \to \infty$

While in the $T \to \infty$ limit the angular coordinates are ignored, in the $\gamma \to \infty$ limit the radial coordinates are ignored. One can show it formally by observing that in this limit the radial node density approaches a delta function—all nodes are placed at the boundary at infinity of the hyperbolic plane, meaning that only angular coordinates determine distances between nodes.

4.1 $T \in (0, \infty)$: Soft spherical random geometric graphs

In this most general case with infinite γ, one can show that the connection probability in Eq. (5) degenerates to

$$p(\theta, \theta') = \frac{1}{1 + \lambda \left(\frac{\Delta\theta}{\pi}\right)^{\beta}}, \tag{14}$$

where $\Delta\theta = \pi - |\pi - |\theta - \theta'||$ is the angular distance between the two nodes as before, while λ is a parameter controlling the average degree $\bar{k}$ in the network, analogous to R in the regimes with finite γ. Without loss of generality we can set $\theta = 0$, so that the expression for $\bar{k}$ is

$$\bar{k} = \frac{N}{\pi} \int_0^\pi \frac{1}{1 + \lambda \left(\frac{\theta'}{\pi}\right)^{\beta}}\, d\theta' = N\, {}_2F_1(1, T; T + 1; -\lambda), \tag{15}$$

where ${}_2F_1$ is the Gauss hypergeometric function, and $T = 1/\beta$. In the special case with $T = 1$, the last expression simplifies to

$$\frac{\bar{k}}{N} = \frac{\log(1 + \lambda)}{\lambda}. \tag{16}$$

If $T \neq 1$, the hypergeometric function in Eq. (15) cannot be evaluated using the GSL library, because the ${}_2F_1$ evaluation in the library is implemented only for the case where the fourth argument of the function ($-\lambda$ in Eq. (15)) is between

-1 and 1, while for sufficiently large $N/\bar{k}$, λ is always larger than 1 in Eq. (15). To avoid this difficulty, the following transformation is used [2]:

$$
{}_2F_1(1,T;T+1;-\lambda) = \frac{1}{\lambda+1}\frac{T}{T-1}\,{}_2F_1(1,1;2-T;\frac{1}{\lambda+1}) + \frac{1}{\lambda^T}\frac{\pi T}{\sin \pi T} = \frac{\bar{k}}{N}, \tag{17}
$$

If $T > 1$ is an integer, the second term in (17) diverges due to the sin function in the denominator, while the first term diverges because the third parameter of the ${}_2F_1$ function is a non-positive integer. Hence, for integer values of temperature $T > 1$, their value is approximated by $T + \epsilon$, where ϵ is set to 10^{-6} by default. The error caused by this approximation is negligible. Equation (17) (or (16) if $T = 1$) is then numerically solved for λ using the bisection method, yielding the target value of $\bar{k}$ in Eq. (15).

4.2 Limit $T \to 0$: Spherical random geometric graphs

One can see from Eq. (17) that the solution for λ at small $T \ll 1$ scales with $N/\bar{k}$ as $\lambda = (N/\bar{k})^\beta$, $\beta = 1/T$. Therefore for $\beta \gg 1$ the connection probability in Eq. (14) can be written as

$$
p(\theta, \theta') = \frac{1}{1 + \left(\frac{N}{\bar{k}}\frac{\Delta\theta}{\pi}\right)^\beta}, \tag{18}
$$

which in the $\beta \to \infty$ limit becomes

$$
p(\theta, \theta') = \Theta\left(1 - \frac{N\Delta\theta}{\bar{k}\pi}\right), \tag{19}
$$

meaning that two nodes are connected if the angular distance $\Delta\theta$ between them is smaller than $\pi\bar{k}/N$,

$$
\Delta\theta < \pi\frac{\bar{k}}{N}, \tag{20}
$$

or they are not connected otherwise. This connectivity threshold ensures that the expected average degree in the network is $\bar{k}$.

4.3 Limit $T \to \infty$: Erdős-Rényi graphs

In this most degenerate regime, both angular and radial coordinates are completely ignored. This regime is formally achieved by keeping both α and ζ finite while letting $T \to \infty$. One can then show that the connection probability in Eq. (5) degenerates to

$$
p(x) = \frac{1}{1 + \frac{N}{\bar{k}}}, \tag{21}
$$

which for sparse graphs with $\bar{k} \ll N$ tends to $p(x) = \bar{k}/N$, i.e., the connection probability in classical (Erdős-Rényi) random graphs.

5 Summary

This section summarizes the implementation details of the six regimes of the model. In all the six cases, the angular and radial node coordinates are assigned using the very similar procedures described in Section 2. The main differences are only in the connection probabilities $p(x)$, and in how the parameters controlling the average degree (e.g., R or λ) are computed.

γ \ T	0	$(0,\infty)$	∞
$[2,\infty)$	1	2	3
∞	4	5	6

Table 2: Regimes of the model with numbers referring to the list below.

1. **Hyperbolic RGGs**

   Connection probability: Eq. (9): each pair of nodes is connected if and only if the hyperbolic distance x between the nodes is smaller than radius R.

   Solve R: Numerical integration of Eq. (8) with $p(x)$ in Eq. (9).

2. **Soft hyperbolic RGGs**

   Connection probability: Eq. (5).

   Solve R: Numerical integration of Eq. (8) with $p(x)$ in Eq. (5).

3. **Soft configuration model**

   Connection probability: Eq. (12).

   Solve R: Numerical integration of Eq. (13).

4. **Spherical RGGs**

   Connection probability: Each pair of nodes is connected if and only if the angular distance between them is smaller than the threshold in Eq. (20).

5. **Soft spherical RGGs**

   Connection probability: Eq. (14).

   Solve λ: Numeric solution of Eq. (17).

6. **Erdős-Rényi**

   Connection probability: Eq. (21).

References

[1] D. Krioukov, F. Papadopoulos, M. Kitsak, A. Vahdat and M. Boguñá,
Phys Rev E 82, 036106. (2010)

[2] R. C. Forrey,
J Comp Phys 137, 79. (1997)