Duality between static and dynamic networks

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Motivation

- Real networks are growing, but static network models exist and are widely used
  - Simpler
  - More tractable
- In a recent work, a minimalistic growing extension of a “very static” model describes remarkably well the growth dynamics of some real networks (*F. Papadopoulos et al., Nature, 489:537, 2012*)
- Informally:
  - One camp: there ***cannot be*** any connection
  - Another camp: there ***should be*** some connection
    - Preferential attachment versus configuration model: ***what is the difference???***

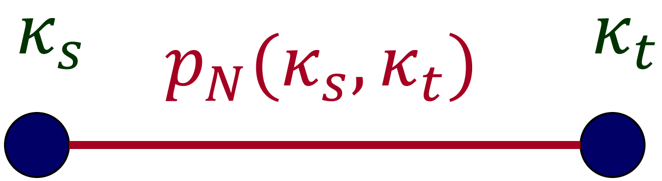
Graph ensemble

- Set of graphs $\mathcal{G}$
- With probability measure $P(G)$,

$$\sum_{G \in \mathcal{G}} P(G) = 1$$

Random graphs with hidden variables

- Given N nodes
- Assign to each node a random variable κ drawn from distribution $\rho_N(\kappa)$
- Connect each node pair (s,t) with probability $p_N(\kappa_s,\kappa_t)$

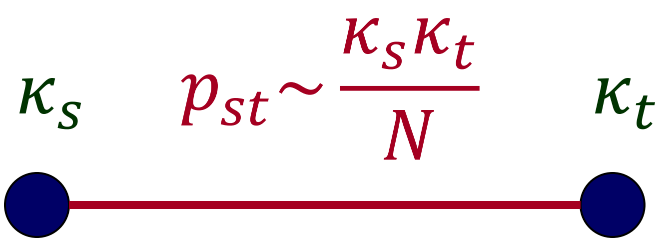


Example: Classical random graphs

- Given N nodes
- Hidden variables: None
- Connection probability: $p_N(\kappa_s, \kappa_t) = p$
- Results:
  - Degree distribution: Poisson ($\lambda = pN$)
  - Clustering: Weak

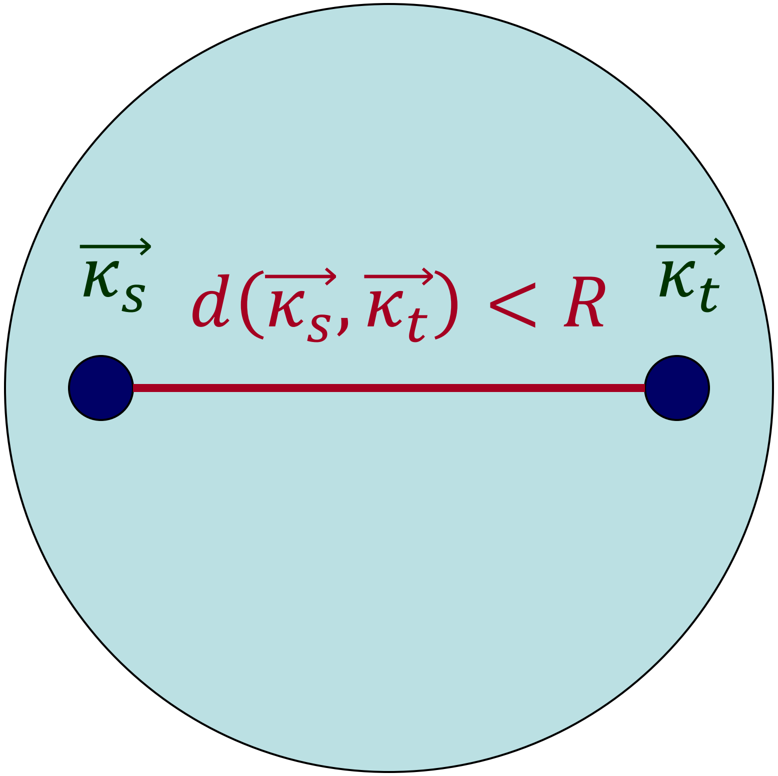
Example:
Soft configuration model

- Given N nodes
- Hidden variables:
  Expected degrees κ
  $$\rho(\kappa) \sim \kappa^{-\gamma}$$
- Connection probability:
  $$p_N(\kappa_s, \kappa_t) = \frac{\kappa_s \kappa_t}{\bar{k} N}$$
- Results:
  - Degree distribution:
    Power law ($P(k) \sim k^{-\gamma}$)
  - Clustering:
    Weak



Example: Random geometric graphs

- Given N nodes
- Hidden variables:
  Node coordinates $\vec{\kappa}$ in a space
  $\rho_N(\vec{\kappa})$: uniform in the space
- Connection probability:
  $p_N(\overrightarrow{\kappa_s}, \overrightarrow{\kappa_t}) = \Theta(R - d_{st})$
- Results:
  - Clustering:
    Strong
  - Degree distribution:
    - Euclidean or spherical space:
      Poisson
    - Hyperbolic or de Sitter spaces:
      Power law



Probability measure

- Let
  - $G = \{a_{st}\}$ the adjacency matrix
  - $\vec{\kappa} = \{\kappa_1, \kappa_2, \dots, \kappa_N\}$ a hidden variable assignment
  - $p_{st} = p(\kappa_s, \kappa_t)$ the connection probability matrix
- Then
  - $P(G|\vec{\kappa}) = \prod_{s<t} p_{st}^{a_{st}} (1 - p_{st})^{1-a_{st}}$
  - $\rho_N(\vec{\kappa}) = \prod_{t=1}^{N} \rho_N(\kappa_t)$
  - $P(G) = \int P(G|\vec{\kappa}) \rho_N(\vec{\kappa}) \, d\vec{\kappa}$

Static versus dynamic graphs with hidden variables

- Static ensemble $\mathcal{G}_S$
  - Given N nodes
  - Assign to each node $t = 1,2, \ldots, N$ a random variable κ_t drawn from distribution $\rho_N(\kappa_t)$
  - Connect each node pair (s, t) with probability $p_N(\kappa_s, \kappa_t)$
- Dynamic ensemble $\mathcal{G}_D$
  - For each new node $t = 1,2, \ldots$
  - Assign to t a random variable κ_t drawn from distribution $\rho_t(\kappa_t)$
  - Connect t to each existing node s with probability $p_t(\kappa_s, \kappa_t)$

Equivalence

- Let dynamic $t = 1,2, \ldots, N$
- Two ensembles $\mathcal{G}_S$ and $\mathcal{G}_D$ are identical ($\mathcal{G}_S = \mathcal{G}_D$) if $P_S(G) = P_D(G)$ for all $G \in \mathcal{G}$
- Distributions $P_S(G) = P_D(G)$ if
  - $\rho_S(\vec{\kappa}) = \rho_D(\vec{\kappa})$ and
  - $p_S(\kappa_s, \kappa_t) = p_D(\kappa_s, \kappa_t)$
- The difference is
  - $\rho_N(\kappa_t)$ (static) versus $\rho_t(\kappa_t)$ (dynamic)
  - $p_N(\kappa_s, \kappa_t)$ (static) versus $p_t(\kappa_s, \kappa_t)$ (dynamic)

Weak equivalence

- Ensembles $\mathcal{G}_S$ and $\mathcal{G}_D$ are weakly equivalent if $\mathcal{G}_S = \mathcal{G}_D$ for some N
- The simplest example
  - Dynamic $\rho_t(\kappa_t)$ is equal to static $\rho_N(\kappa_t)$
  - Dynamic $p_t(\kappa_s, \kappa_t)$ is equal to static $p_N(\kappa_s, \kappa_t)$
  - The difference is only in node labeling

Strong equivalence

- Ensembles $\mathcal{G}_S$ and $\mathcal{G}_D$ are strongly equivalent if $\mathcal{G}_S = \mathcal{G}_D$ for any N
- In this case the connection probability cannot depend on graph size N
  - $p_N(\kappa_s, \kappa_t) = p_t(\kappa_s, \kappa_t) = p(\kappa_s, \kappa_t)$
- The simplest example
  - $\rho_N(\kappa_t) = \rho_t(\kappa_t) = \rho(\kappa_t)$
  - Graphs are dense because average degree $\bar{k} = N \iint \rho(\kappa) p(\kappa, \kappa') \rho(\kappa') \, d\kappa \, d\kappa' = O(N)$
- In sparse strongly equivalent ensembles, the connection probability does not depend on graph size, but the hidden variable distribution does

Example: Classical random graphs

- Ensemble $\mathcal{G}_{N,p}$ is strongly equivalent and dense since $\bar{k} = Np$
  - The growing definition is: connect new nodes t to existing nodes $s < t$ with probability p
- An attempt to fix: ensemble $\mathcal{G}_{N,\bar{k}}$:
  - Static: connect all node pairs with probability $p = \frac{\bar{k}}{N}$
    - The ensemble is $\mathcal{G}_{N,p}$
    - The degree distribution is Poisson
  - Dynamic: connect new nodes t to existing nodes $s < t$ with probability $p = \frac{\bar{k}}{t}$
    - The ensemble is not $\mathcal{G}_{N,p}$
    - The degree distribution is exponential
- The ensembles are very different because the connection probability depends on graph size

Example:
Soft configuration model

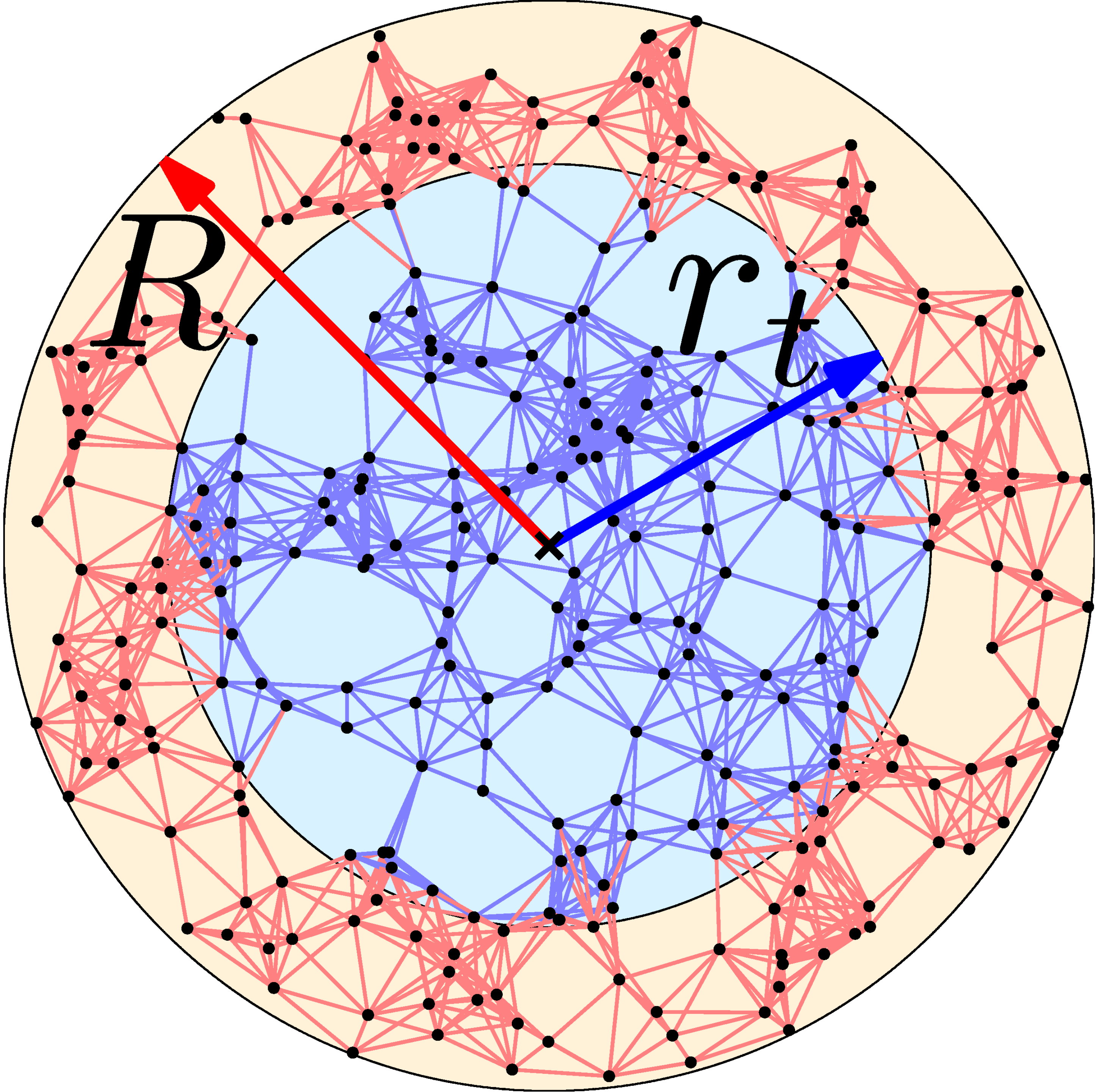
- Strong equivalence in the dense case (graphons)
- No strong equivalence in the sparse case since the connection probability $p_{st} \sim \frac{\kappa_s \kappa_t}{N}$ depends on graph size

Example: Preferential attachment

- Weak equivalence in the soft formulation (*M.Boguna et al.*, *PRE 68:36112*, *2003*)
- No strong equivalence since the probability that new node t connects to existing node s $p_{st} \sim \frac{k_s}{\sum_s k_s}$ depends on graph size
- Needless to say, $P(G)$ is unknown

Example:
Random geometric graphs

- Strong equivalence in sparse graphs!
- Connection probability $p_{st} = \Theta(R - d_{st})$ does not depend on graph size
- The distribution of new node coordinates $\rho_t(\kappa_t)$ is such that $\rho_S(\vec{\kappa}) = \rho_D(\vec{\kappa})$ because both static and dynamic constructions implement the same (Poisson) point process



R
r_t

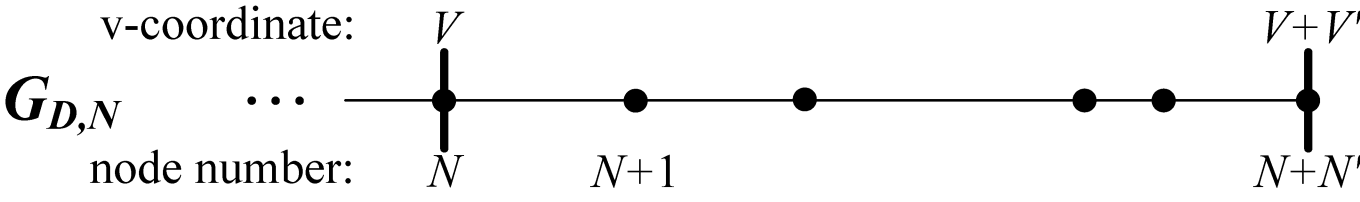
Poisson point processes behind random geometric graphs

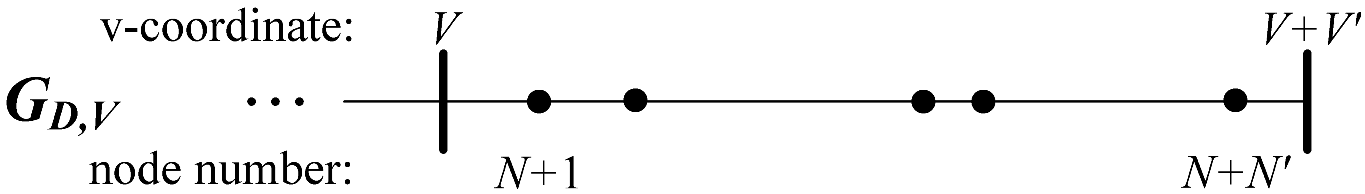
- Let r_t be the radial coordinate of node t
- Then $V_t = |B(0, r_t)|$ is its volume coordinate
- Points $\{V_t, t = 1,2, \ldots\}$ is a PPP on $\mathbb{R}_+$

Two pairs of identical ensembles

- Fixed-volume pair
  - Static ensemble $\mathcal{G}_{S,V}$
    - Given V, sample N from the Poisson distribution with mean δV, where δ is the point density (PPP rate)
    - Sample N points on $[0, V]$
  - Dynamic ensemble $\mathcal{G}_{D,V}$
    - Given ΔV, sample N from the Poisson distribution with mean $\delta \Delta V$, where δ is the point density (PPP rate)
    - Sample N points on $[V, V + \Delta V]$
- Fixed-size pair
  - Static ensemble $\mathcal{G}_{S,N}$
    - Given N, sample V from the Gamma distribution with rate δ and shape N
    - Sample $N - 1$ points on $[0, V]$
  - Dynamic ensemble $\mathcal{G}_{D,N}$
    - Given ΔN, sample ΔV from the Gamma distribution with rate δ and shape ΔN
    - Sample $\Delta N - 1$ points on $[V, V + \Delta V]$

Two pairs of identical ensembles

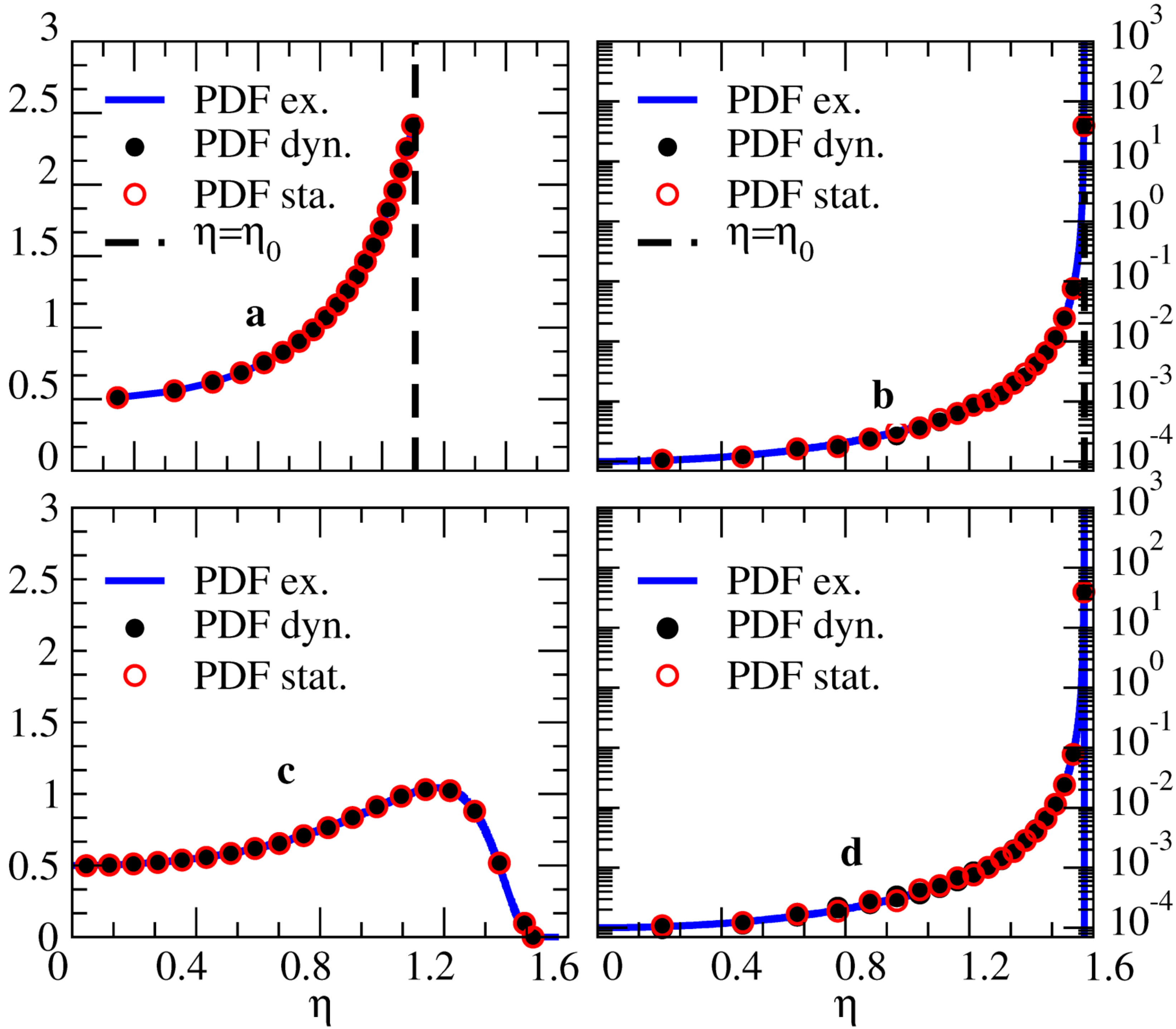




$$\begin{array}{ccc} \mathcal{G}_{S,V} & = & \mathcal{G}_{D,V} \\ \wr & & \wr \\ \mathcal{G}_{S,N} & = & \mathcal{G}_{D,N} \end{array}$$

Final example: de Sitter causal sets

- Random geometric graphs in de Sitter spacetime
  - Angular coordinates θ are as in all spherically symmetric random geometric graphs, i.e., sampled from the uniform distribution on a unit sphere $\mathbb{S}^d$
  - “Radial” (conformal time) coordinates η are sampled from $\rho(\eta) \sim \sec^{d+1} \eta$
  - Nodes are connected if $\Delta\eta > \Delta\theta$
- The resulting graph ensembles are
  - Strongly equivalent
  - Sparse
  - Have power-law distributions of node degrees with exponent $\gamma = 2$
  - Have strong clustering



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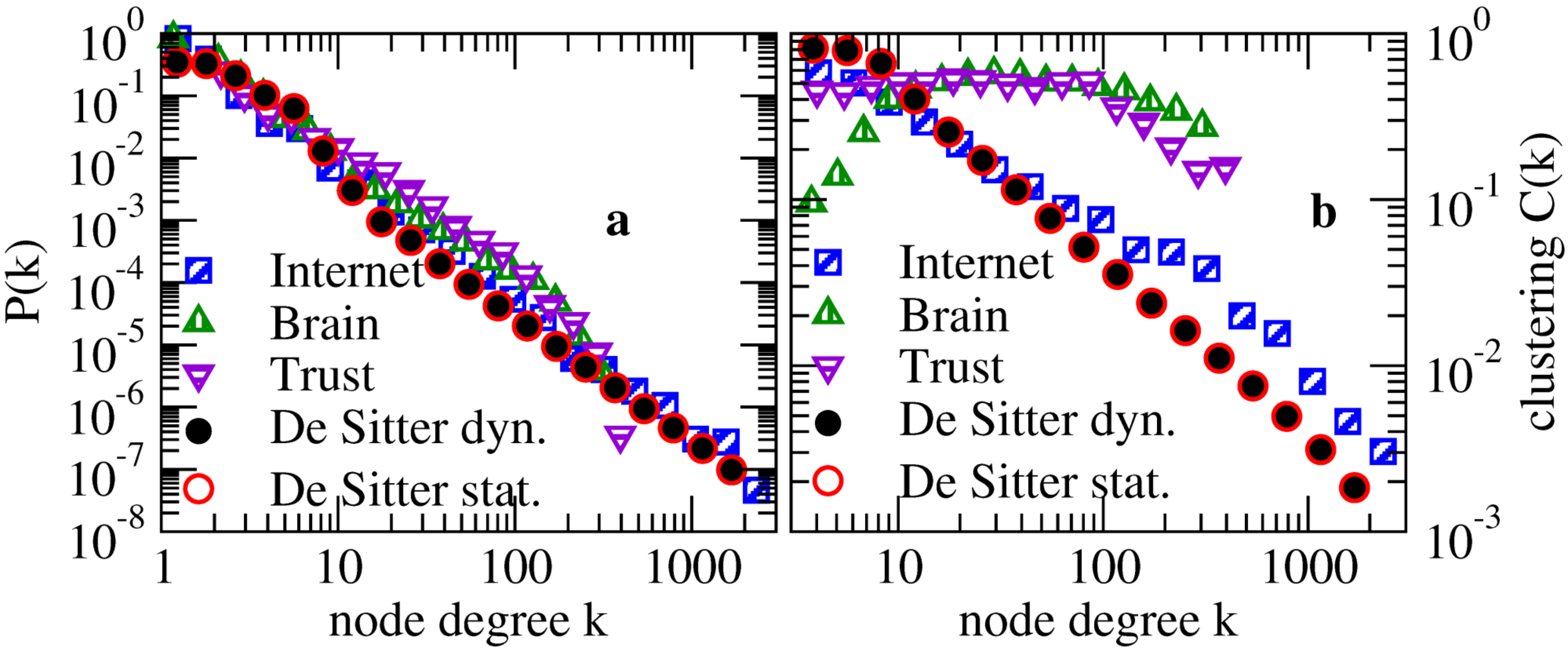

P(k)
a
Internet
Brain
Trust
De Sitter dyn.
De Sitter stat.
node degree k
b
clustering C(k)

Conclusions

- There does exist a very simple connection between equilibrium and non-equilibrium approaches to network analysis
- Some real networks appear to be “close” to graph ensembles that admit dual, equilibrium and growing, description
- This justifies applications of powerful equilibrium tools to the analysis of real networks
- Does nature optimize?
- If the least action principle applies,
  then what is the action?

- D. Krioukov and M. Ostilli
  Duality between equilibrium and growing networks,
  PRE, v.88, p.022808, 2013